Fluctuation dissipation theorems (review)

- From Langevin approach of Brownian Motion $F(t) = \sqrt{2k_BT\gamma}\,\Gamma(t)$

$$\langle F(t+t')F(t)\rangle = 2k_BT\gamma\,\delta(t')$$

integrating over t' yields:

$$2k_BT\gamma = \int_{-\infty}^{\infty} \langle F(t+t')F(t)\rangle dt$$

Velocity Fluctuations:

$$\lim_{t\to\infty} \langle v(t)v(t+t')\rangle = \frac{c\tau}{2} e^{-t'/\tau}$$

where c,τ are coefficients of the O-U process. $\left(c = \frac{2}{D}\left(\frac{kT}{m}\right)^2 \wedge \tau = \frac{m}{\gamma}\right)$

$$\Leftrightarrow \boxed{D = \int_0^{\infty} \langle v(t)v(t+t')\rangle\, dt' = \frac{k_BT}{m\gamma}}$$

FLUCTUATION DISSIPATION FORMULA (EINSTEIN RELATION)

→ From autocorrelation spectrum of velocity fluctuations one can derive particle damping (for known Temp. T)

- For the case of Johnson Noise we have: (V: voltage noise)

$$\langle V(t)V(t+t')\rangle = 2k_BTR\,\delta(t')$$

and since (I: current)

$$\langle I(t)I(t+t')\rangle = \frac{c\tau}{2} e^{-t/\tau}$$

(Ornstein-Uhlenbeck with $c = \frac{k_BTR}{L}$; $\tau = \frac{R}{L}$)

Thus:

$$\boxed{R^{-1} = \frac{1}{k_BT}\int_0^{\infty} \langle I(t)I(t+t')\rangle dt'} \quad (*)$$

FORMULA IS CALLED "CONDUCTANCE" FORMULA.
IT relates current fluctuations to the resistance.

→ one application: "Low-temperature Thermal Noise Measurement" Shore, Rev. Sci. Instr. 1959)
→ Temperature measurements.
$\langle V^2(f)\rangle = 4k_BTR\,\delta f$ Noise power in f..f+df.

Generalized fluctuation dissipation relation

The damping can also be frequency dependent (e.g. electrical circuits $R(\omega)$).
How does this modify the relation (*) COND. FORMULA?

Answer was given by Callen & Welton "irreversibility & generalized Noise" and Kubo "Fluctuation Diss. Theorem" 1966:

$$\boxed{\dot{v}(t) = -\int_0^t \gamma(t-s)v(s)ds + \frac{F(t)}{m}}$$

$\gamma(t)$: retarded friction term

Lecture 4.

Proof of the G.FDT

GENERALIZED FLUCTUATION DISSIPATION RELATION (KUBO)

$$\dot v(t) + \int_0^t \gamma(t-s)\, v(s)\, ds = F_L(t)/m$$ (*) $F(t)$: Langevin force

Fourier transform: def. $x(\omega) = \int x(t)\, e^{-i\omega t}$ and $x(t) = \int x(\omega) \frac{d\omega}{2\pi} e^{+i\omega t}$

instantaneous damping $\gamma(t-s) = \delta(t-s)\,\gamma$

implies

$$(i\omega + \gamma[\omega])\, V(\omega) = F(\omega)/m$$

define $\boxed{\chi(\omega) \equiv \frac{1}{m(i\omega + \gamma[\omega])}}$ SUSCEPTIBILITY OR ADMITTANCE and: $V(\omega) = \chi(\omega) \cdot F(\omega)$ [Eq (2)]

$$\boxed{\langle V(t) V(0)\rangle = k_B T\, \chi(t)} \quad (1)$$

Proof: time domain solution (*) is given by using Eq (2):

$$V(t) = \underbrace{m V(0)\, \chi(t)}_{\text{homogeneous part}} + \underbrace{\int_0^t \chi(t-s)\, F(s)\, ds}_{\text{inhom. part}} \quad \wedge \quad \chi(t) = e^{-\gamma t}\frac{1}{m}$$

$$\Rightarrow \langle v(t) v(0)\rangle = m \langle V(0)^2\rangle\, \chi(t) + \int_0^t \chi(t-s)\, \underbrace{\langle v(0) F(s)\rangle}_{=0}\, ds$$

$=0$ since Force & velocity are not correlated at time $t=0$

$$\overset{(\ast)}{=} m\langle V(0)^2\rangle\, \chi(t)$$

(Boltzmann)

$$= \frac{k_B T}{m} \cdot m\, \chi(t) = k_B T\, \chi(t)$$

NOTE: in general: $\chi(t) \overset{(\ast)}{=} \frac{\langle v(t) v(0)\rangle}{m\langle v^2\rangle} \Rightarrow \chi(\omega) = \frac{1}{m\langle v^2\rangle} \int_0^\infty \langle v(t) v(0)\rangle e^{-i\omega t} dt$

$$\boxed{\int_{-\infty}^{\infty} \langle F_L(t) F_L(0)\rangle e^{-i\omega t} dt = 2 k_B T\, \mathrm{Re}\{\chi^{-1}(\omega)\}} \quad (2)$$ GENERALIZED FLUCTUATION DISSIPATION RELATION

Proof: $\langle |V(\omega)|^2\rangle = |\chi(\omega)|^2 \langle |F(\omega)|^2\rangle$

thus: $\int \langle v(t) v(0)\rangle e^{-i\omega t} dt = |\chi(\omega)|^2 \int \langle F(t) F(0)\rangle e^{-i\omega t} dt$

Due to causality $\langle v(t) v(0)\rangle$ is an even function of time: i.e. $C(T) = C(-T)$

$$\int_{-\infty}^{\infty} \langle v(t) v(0)\rangle e^{-i\omega t} dt = 2\,\mathrm{Re}\left\{\int_0^\infty \langle v(t) v(0)\rangle e^{-i\omega t} dt\right\}$$

Thus: $$|\chi(\omega)|^2 \int_{-\infty}^{\infty} \langle F(t) F(0)\rangle e^{-i\omega t} dt = 2\,\mathrm{Re}\left(\int_0^\infty \langle v(t) v(0)\rangle e^{-i\omega t} dt\right)$$

using Eq. 1

$$= 2\,\mathrm{Re}\,[k_B T\, \chi(\omega)]$$

$$\Leftrightarrow \int_{-\infty}^{\infty} \langle F(t) F(0)\rangle e^{-i\omega t} dt = 2 k_B T \frac{\mathrm{Re}\,[\chi(\omega)]}{|\chi(\omega)|^2} = 2 k_B T\, \mathrm{Re}\left\{\frac{1}{\chi(\omega)}\right\}$$

Lecture 4

$$\int_0^\infty \langle F_L(t) F_L(0)\rangle e^{i\omega t}\, dt = k_B T\, \gamma(\omega) \hat{=} \frac{k_B T}{m} \mathrm{Re}\{ \chi^{-1}[\omega]\}$$

GENERALIZED FDT.

Proof:

Since $\chi[\omega]^{-1} = m(i\omega + \gamma[\omega])$ it implies that

$$\mathrm{Re}\{\chi^{-1}[\omega]\} = m \cdot \mathrm{Re}\{\gamma[\omega]\}$$

using Eq.(2) - yields:

$$\int_{-\infty}^{\infty} \langle F(t) F(0)\rangle e^{i\omega t}\, dt = 2 k_B T m\, \mathrm{Re}\{\tilde{\gamma}(\omega)\}$$

Using causality and stationarity arguments for $F(t)$ ... ensures that $\langle F(t)F(0)\rangle$ is even in time t.

$$\int_{-\infty}^{\infty} \langle F(t) F(0)\rangle e^{i\omega t}\, dt = 2\, \mathrm{Re} \int_0^\infty \langle F(t) F(0)\rangle e^{i\omega t}$$

Thus:

$$2\,\mathrm{Re}\int_0^\infty \langle F(t) F(0)\rangle e^{i\omega t}\, dt = 2 k_B T\, \mathrm{Re}\{\gamma(\omega)\}$$

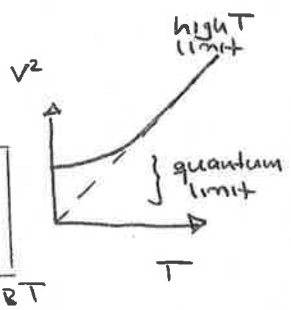
Theorem in complex analysis: if two analytical functions have equal real parts, the functions must be equal.

Since: If ~~the~~ system is stable, then $\chi(\omega)$ must be analytical in $\mathrm{Re}\{\omega\} < 0$ complex plane.

Likewise if $F(t)$ is physical, F must be analytical in $\mathrm{Re}\{\omega < 0\}$

$$\Rightarrow \quad \boxed{k_B T\, \gamma(\omega) = \int_0^\infty \langle F(t) F(0)\rangle e^{-i\omega t}\, dt}$$

Finally, the above Generalized F.D.T is not correct in the limit of $T \to 0$. This necessitates a quantum mechanical treatment (Callen & Welton 1951) quoting from original paper



For:

$$(*) \quad \boxed{\langle V^2\rangle = \frac{2}{\pi}\int_0^\infty R(\omega)\Big[\underbrace{\frac{\hbar\omega}{2} + \hbar\omega\big[e^{\hbar\omega/kT} - 1\big]^{-1}}_{E(\omega,T)}\Big]}$$

as $T \gg \hbar\omega$ expression is $\to k_B T$

classical limit: $k_B T \gg \hbar\omega$ leads to: $E(\omega, T)$: mean energy of an harmonic oscillator (freq. ω) at temp. T.

$$\langle V^2\rangle = \frac{2}{\pi}\int_0^\infty R(\omega)\, k_B T$$

NOTE

(*) Definition of Fourier transform uses $1/2\pi$ in time to frequency
→ Results differ.

Lecture 4

Fokker Planck Equation

Goal: derive Fokker Planck equation from stochastic differential equation, ie show that: any stochastic process

$$dx(t) = a(x,t)\,dt + \underbrace{T(t)\,dt}\,D(x,t) \qquad T(t) = \lim_{dt\to 0} N(0,\tfrac{1}{dt})$$

$dW \equiv T(t)dt$: Wiener increment

can be written for the Probability* $p(x_0,t_0 \mid x,t)$ as:

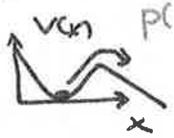
Forward F.P.

$$\boxed{\partial_t p(x_0 t_0 \mid x,t) = -\frac{d}{dx}\left[a(x,t)\,p(x,t \mid x_0,t_0)\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[D(x,t)\,p(x_0 t_0 \mid x,t)\right]}$$

(FOKKER PLANCK EQUATION, ordinary diff eqs)

initial conditions $p(x_0 t_0 \; x,t) = \delta(x_0 - \bar{x}_0, t_0 - \bar{t}_0)$

Application: Kramer's escape rate

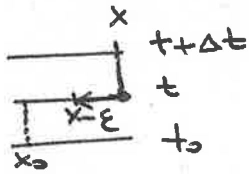


* conditional probabilities.

Quick review: Chapman-Kolmogorov Equation. (required for proof)

(I) $$\underbrace{p(x_0,t_0}_{\text{initial condition (fixed)}} \mid x,t+\Delta t) = \int_{-\infty}^{\infty} d\varepsilon\; p(x_0,t_0 \mid x-\varepsilon,t)\,p(x-\varepsilon,t \mid x,t+\Delta t)$$

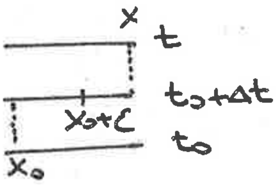
Graphically:



or the reverse equation:

(II) $$p(\underbrace{x_0,t_0}_{\text{varying}} \mid \underbrace{x,t}_{\text{fixed final state}}) = \int_{-\infty}^{\infty} d\varepsilon\; p(x_0,t_0 \mid x_0+\varepsilon, t_0+\Delta t)\,p(x_0+\varepsilon, t_0+\Delta t \mid x,t)$$

Graphically:



Note: we will need these two forms of the Chapman-Kolmogorov equation for the derivation of the FORWARD and BACKWARD Fokker Planck Equation.

For a homogeneous process: (time invariant to translation)

$$p(x_0,t_0 \mid x,t) = p(x_0, t_0 - t \mid x, 0)$$

thus:

$$\partial_t p(x_0,t_0 \mid x,t) = \partial_{t_0} p(x_0,t_0 \mid x,t) = -\partial_t\, p(x_0, t_0 - t \mid x,t)$$

Fokker Planck Chapman Colmogorov eqs

Review on Absolute and conditional probabilities and absolute probabilities.

$W(x_1,t_1; x_2,t_2; \dots x_n,t_n) \cdot dx_1 \dots dx_n$ } Probability to find particle in the intervals $x_1 \dots dx_1 + x_1$ at times t_1.

Properties:
(i) $W > 0$ (ii) $\int dx_1 \dots dx_n W = 1$ (iii) symmetric under permutation
(iv) $\int dx_n W(\dots x_n, t_n) = W(x_{n-1}, t_{n-1})$

Stationary process (i) $W(x_1,t_1; x_2 t_2) = W(x_1, 0; x_2, t_2 - t_1)$
(ii) $W(x_1,t_1) = W(x_1)$

$\lim_{t-t_0 \to \infty} P(x,t|x_0,t_0) = P_S(x)$
STATIONARY MARKOV STATE DENSITY FUNCT.

Conditional probability:

$$P(\underbrace{x_1 t_1 \dots x_k t_k}_{\text{knowing it was there before}} | \underbrace{x_{k+1} t_{k+1} \dots x_n, t_n}_{\text{Probability to find particle here}}) = \frac{W(x_1,t_1 \dots x_n,t_n)}{W(\dots x_k, t_k)}$$

Brownian motion $a(v) = -\gamma v$, $b(v) = 2\gamma^2 D$ ⇒ we already encountered the conditional probability. Gillespie Markov Jump approach: O-U.

$P(v_1,t_1|v_2,t_2=t_1) = \delta(v_2 - v_1)$ INITIAL CONDITION

$$\boxed{P(v_1 t_1|v_2,t_2) = \sqrt{\frac{m\beta}{2\pi}} \times \frac{1}{\sqrt{1-e^{-2\gamma(t_2-t_1)}}} \exp\left[-\beta\frac{m}{2}\frac{v_2 - v_1 e^{-\gamma(t_2-t_1)}}{1-e^{-2\gamma(t_2-t_1)}}\right]}$$

initial condition decays

In the long time limit

$$\boxed{\lim_{t_2\to\infty} P(v_1 t_1|v_2 t_2) = \sqrt{\frac{m\beta}{2\pi}}\, e^{-\frac{1}{2}\beta m v^2} = P(v) \hat{=} W(v)}$$

STATIONARY DISTRIBUTION (MAXWELL BOLTZMANN)

Any problem can be written into a F-P equation. The FP can also be generalised to higher dimension

References
- Proof: Gillespie Book
- See also KUBO. Ch. 2.4

Fokker Planck & Kramers Moyal Expansion

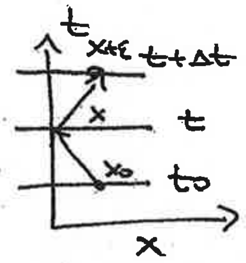
$$P(x_0,t_0|x,t+\Delta t) = \int_{-\infty}^{\infty} d\varepsilon \; P(x_0,t_0|x-\varepsilon,t)\, P(x-\varepsilon,t|x,t+\Delta t) \quad (1)$$

CHAPMAN KOLMOGOROV

$t_0 < t < t+\Delta t$
"Forward"

Next define

$$\boxed{f(x) \equiv P(x_0,t_0|x,t)\, P(x,t|x+\varepsilon,t+\Delta t)}$$



Taylor:

$$\boxed{f(x-\varepsilon) = f(x) + \sum_{k=1}^{\infty} \frac{(-\varepsilon)^k}{k!} \frac{\partial^k}{\partial x^k} f(x)}$$

Substitute:

$$f(x-\varepsilon) = P(x_0 t_0|x-\varepsilon,t)\, P(x-\varepsilon,t|x,t+\Delta t) \quad (2)$$

Thus:

$P(x_0,t_0|x,t+\Delta t) = \int_{-\infty}^{\infty} f(x-\varepsilon)\,d\varepsilon$ since (1) & (2) are identical.

Use Taylor expansion:

$$P(x_0,t_0|x,t+\Delta t) = \int_{-\infty}^{\infty} d\varepsilon \left[f(x) + \sum_{k=1}^{\infty} \frac{(-\varepsilon)^k}{k!} \frac{\partial^k}{\partial x^k} f(x) \right]$$

$$= \underbrace{\int_{-\infty}^{\infty} d\varepsilon\, P(x_0,t_0|x,t)\, P(x,t|x+\varepsilon,t+\Delta t)}_{P(x_0,t_0|x,t)} \quad \text{since: } \int_{-\infty}^{\infty} d\varepsilon\, P(x,t|x+\varepsilon,t+\Delta t) = 1$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{-\infty}^{\infty} d\varepsilon\, \frac{\partial^k \varepsilon^k}{\partial x^k} \left[P(x_0,t_0|x,t)\, P(x,t|x+\varepsilon,t+\Delta t) \right]$$

Pull out of integral (interchange differentiation & integration)

Thus:

$$\boxed{\frac{P(x,t+\Delta t|x_0,t_0) - P(x,t|x_0,t_0)}{\Delta t} = \frac{1}{\Delta t} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} \int_{-\infty}^{\infty} d\varepsilon\, \varepsilon^k P(x_0,t_0|x,t) \times P(x,t|x+\varepsilon,t+\Delta t)}$$

Next analyse terms in Bracket.

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} d\varepsilon\, \varepsilon^k P(x,t|x+\varepsilon,t+\Delta t)$$

$$\Leftrightarrow \boxed{\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} d\varepsilon\, \varepsilon^k P(\varepsilon,\Delta t) = B_n(x,t)}$$

for a stationary process. + homogeneous

Thus: this equals $\boxed{\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\langle x^k \rangle \right]}$

Hence the Equation of $B_n(x,t)$ is identical to the moment of X. This connects to our the

Thus: $$\boxed{\partial_t P(x_0 t_0|x t) = \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \frac{\partial^n}{\partial x^n} \left[B_n(x,t) P(x_0 t_0|x,t) \right]}$$ = FOKKER PLANCK EQUATION DERIV. BY KRAMER–MOYAL EXPANSION

Repeat for e.g. the general case:

$$\boxed{x(t+dt) = x(t) + A(x,t)dt + D^{1/2}(x,t). N(t) \sqrt{dt}}$$ MARKOV PROCESS

here $\langle x \rangle_{dt} = A(x,t)dt$ $\Rightarrow B_1 = A(x,t)$

(use e.g. ITO CALCULUS FOR MOMENTS) $\langle x^2 \rangle_{dt} = D(x,t)(\sqrt{dt})^2$ $\Rightarrow B_2 = D(x,t)$

$\langle x^3 \rangle_{dt} = 0$ $\Rightarrow B_3 = 0$ $(B_n = 0 ;\ n \geq 3)$

Note: $\langle \frac{d}{dt} x^n(t) \rangle = n \langle x^{n-1}(t) A(x,t) \rangle + \frac{n(n-1)}{2} \langle x^{n-2}(t) D(x,t) \rangle$

For a continuous markov process the Kramer Moyal Expansion yields:

(FORWARD) FOKKER PLANCK EQUATION
$$\boxed{\frac{\partial}{\partial t} P(x,t|x_0,t_0) = -\frac{\partial}{\partial x} \left[A(x,t) P(x_0 t_0|x,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[D(x,t) P(x_0 t_0|x,t) \right]}$$

SOLVING THE FP:

Example: Wiener Process $A(x,t) = 0$ $D(x,t) = 2D$

Thus $x(t+dt) = x(t) + D\, N(t) \sqrt{dt}$ (wiener increment)

ie WIENER PROCESS
$$\boxed{\frac{x(t+dt) - x(t)}{dt} = \frac{dx}{dt} = D \cdot N(t) \frac{1}{\sqrt{dt}} \hat{=} 2D . \Gamma(t)\, dt \equiv dW}$$

$P(x_1 t_1 | x_2, t_2)$ can be found by solving $\frac{\partial P}{\partial t} = \left(\frac{1}{2} \frac{\partial}{\partial x^2} P \right) \cdot 2D$

$$\boxed{P(x_1, t_1 | x_2, t_2) = \frac{1}{\sqrt{4\pi D(t_2 - t_1)}} e^{-\frac{(x_2 - x_1)^2}{4D(t_2 - t_1)}}}$$ Transition probability. *1

$W(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$ Distribution → satisfies Chapman Kolmogorov eqs

calculate average $\langle x(t) \rangle = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi D(t-t_0)}} e^{-x^2/4D(t-t_0)} \cdot x = x_0 + A(t - t_0)$

CO VARIANCE
$$C(t_1, t_2) = \langle x(t_1) x(t_2) \rangle = \iint x_1 x_2\, dx_1\, dx_2\, W(x_1 t_1; x_2 t_2)$$
$$= \iint x_1 x_2\, dx_1\, dx_2\, W(x_1 t_1) . P(x_1 t_1 \cdot x_2 t_2)$$ (use expression *1)
$$= \int dx_1 \cdot x_1^2 \cdot W(x_1, t_1) = 2Dt_1$$ (cf. Martin)

→ Fast way to calculate co-variances.

LECTURE 4

Fokker Planck : Ornstein Uhlenbeck:

e.g. velocity of Brown. motion

$a(v) = -\gamma v \qquad b(v) = \frac{2\gamma}{k_B T m}$

$\beta = (k_B T)^{-1}$

$$\boxed{\frac{\partial}{\partial t} P(v,t) = \gamma \frac{\partial}{\partial v}\left[v \cdot P(v,t)\right] + \gamma^2 D \frac{\partial^2}{\partial v^2} P(v,t)}$$

$$P(v_1 t_1 | v_2 t_2) = \sqrt{\frac{m\beta}{2\pi}} \cdot \frac{1}{\sqrt{1 - e^{-2\gamma(t_2 - t_1)}}} \times \mathrm{Exp}\left[-\frac{\beta}{2m^{-1}} \cdot \frac{(v_2 - v_1 e^{-\gamma(t_2 - t_1)})^2}{1 - e^{-2\gamma(t_2 - t_1)}}\right]$$

importantly: satisfies the criterion of Chapman Kolmogorov

$$\boxed{W(v) = \sqrt{\frac{m\beta}{2\pi}}\, e^{-\beta \frac{1}{2} m v^2}} \qquad \beta = (k_B T)^{-1}$$

MAXWELL BOLTZMANN DISTRIBUTION

Alternative Derivation of the FOKKER-PLANCK EQUATION using ITO's LEMMA

Reminder ITO's LEMMA:

$f(x)$ where f: is a function of x, a stoch. Markov process:

$$dx = A(x,t)\,dt + \underbrace{\Gamma(t)\,dt}_{\equiv dW(t) \text{ Wiener Increment}} D(x) \qquad \Gamma(t) = \lim_{dt\to 0} N(0, 1/dt)$$

Proof on Equivalence of S.D.E and Fokker Planck Part. Diff. Eq.:

Ito's Lemma reminder:

$$df = \left(A(x,t)\frac{df}{dx} + \frac{1}{2}D(x)^2\frac{d^2f}{dx^2}\right)dt + D(x,t)\,dW$$

Next: $p(x_0,t_0\,|\,x,t)$ obeys: $\qquad \langle f(x(t))\rangle = \int dx\, f(x)\, p(x_0,t_0|x,t)$

and:

$$\frac{d\langle f(x(t))\rangle}{dt} = \int_{-\infty}^{\infty} dx\, \frac{d}{dt}\left(-f(x)\, p(x_0,t_0|x,t)\right)$$
$$= -\int_{-\infty}^{\infty} dx\, f(x)\frac{d}{dt}p(x_0,t_0|x,t)$$

and:

$$\frac{d\langle f(x(t))\rangle}{dt} = \left\langle \frac{df(x(t))}{dt}\right\rangle = \int_{-\infty}^{\infty} \underbrace{\left\langle \frac{df(x(t))}{dt}\right\rangle}_{\text{ITO LEMMA}} \cdot p(x_0,t_0|x,t)$$
$$= \int_{-\infty}^{\infty}\Big[\underbrace{A(x,t)\frac{df}{dx}}_{\text{1st Term}} + \underbrace{\frac{1}{2}D(x,t)\frac{d^2f}{dx^2}}_{\text{2nd Term}}\Big]\, p(x_0,t_0|x,t)$$

Integrale by parts: alternative: replace $f(x) = \delta(x-\tilde{x})$ and use $\int \frac{d}{dx}\delta(x-\tilde{x})f(x) = \frac{df}{dx}\Big|_{\tilde{x}}$ → avoid IBP.

1st Term:

$$\int_{-\infty}^{\infty} \underbrace{A(x,t)}_{v}\underbrace{\frac{df}{dx}}_{u'}\underbrace{p(x_0,t_0|x,t)}_{v}\,dx = \int_{-\infty}^{\infty} \underbrace{f(x)}_{u}\underbrace{\frac{d}{dx}\left[A(x,t)\,p(x_0,t_0|x,t)\right]}_{v'}dx$$
$$- \underbrace{\left[A(x,t)\,p(x_0,t_0|x,t)\right] f(x)\Big|_{-\infty}^{\infty}}_{\text{vanishes (0)}}$$

Surface Term $\lim_{x\to\pm\infty} p(x_0,t_0|x,t) = 0$

2nd Term

$$\int_{-\infty}^{\infty}\frac{1}{2}D(x,t)\frac{d^2f}{dx^2}p(x_0,t_0|x,t) = -\frac{1}{2}\int\frac{d}{dx}\left[D(x,t)\,p(x_0,t_0|x,t)\right]\frac{df}{dx}\,dx$$
$$+\; p(x_0,t_0|x,t)\,D(x)/2\cdot\frac{df}{dx}\Big|_{-\infty}^{\infty} \qquad \text{Surface Term} \to 0$$

2nd Int. by parts $\qquad = \int + \frac{1}{2}\frac{d^2}{dx^2}\left[D(x,t)\,p(x_0,t_0|x,t)\right)$

$$\Leftrightarrow \int dx\, f(x)\frac{d}{dt}p(x_0,t_0|x,t) = \int dx\Big\{f(x)\Big[\frac{d}{dx}\big(A(x,t)\,p(x_0,t_0|x,t)\big)\Big] + \frac{1}{2}\frac{d^2}{dx^2}\left[D(x,t)\,p(x_0,t_0|x,t)\right]\Big\}$$

$$\Leftrightarrow \boxed{\frac{d}{dt}p(x_0t_0|x,t) = -\frac{d}{dx}\left[A(x,t)\,p(x_0,t_0|x,t)\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[D(x,t)\,p(x_0,t_0|x,t)\right]} \quad \text{F-P Eq.}$$

Backward Fokker Planck Equation: (important for ESCAPE PROBLEMS, ie KRAMER'S rate)

Starting point is backward Kolmogorov Eq (t_0, x_0 variabel):

$$p(x_0,t_0|x,t) = \int_{-\infty}^{\infty} d\varepsilon\, p(x_0,t_0|x_0+\varepsilon,t_0+\Delta t_0)\, \underbrace{p(x_0+\varepsilon, t_0+\Delta t_0|x,t)}_{\text{corresponds to } u(x_0+\varepsilon)}$$

Define function: $\underline{u(x_0) = p(x_0,t_0|x,t)}$

and $u(x_0+\varepsilon) = u(x_0) + \sum_u \varepsilon^u \frac{1}{u!}\frac{\partial^u}{\partial x_0^u} u(x_0)$ expand around ε in x_0

next:

$$p(x_0 t_0|x,t) = \int_{-\infty}^{\infty} d\varepsilon\, p(x_0,t_0|x+\varepsilon,t_0+\Delta t_0)\cdot\Big\{ p(x_0,t_0+\Delta t_0|x,t) + \varepsilon^u \frac{\partial^u}{\partial x_0^u}\big(p(x_0,t_0+\Delta t_0|x,t)\big)\Big\}$$

Key point to observe: in backward Kramer's Moyal expansion

$$\int_{-\infty}^{\infty} \varepsilon^u \frac{1}{u!}\frac{\partial^u}{\partial x_0^u}\big(p(x_0,t_0+\Delta t_0|x,t)\big)\cdot p(x_0,t_0|x_0+\varepsilon,t_0+\Delta t_0)$$

acts only on 1st term.

$$= \frac{\partial^u}{\partial x_0^u}\cdot p(x_0,t_0+\Delta t|x,t)\cdot \underbrace{\int_{-\infty}^{\infty} \varepsilon^u \frac{1}{u!} p(x_0,t_0|x_0+\varepsilon,t_0+\Delta t_0)}_{\text{Expression yields again the moments of stoch. diff. eqs.}}$$

Take again limit and use $\int_{-\infty}^{\infty} d\varepsilon\, p(x_0,t_0|x+\varepsilon,t_0+\Delta t_0) = 1$

$$\lim_{\Delta t_0 \to 0} \frac{p(x_0 t_0|x,t) - p(x_0,t_0+\Delta t_0|x,t)}{\Delta t_0} = -\frac{\partial}{\partial t_0} p(x_0,t_0|x,t)$$

This yields for a stochastic process

$$-\frac{\partial}{\partial t_0} p(x_0,t_0|x,t) = \sum_{u=1}^{\infty} \frac{1}{u!} B(x_0,t_0) \frac{\partial^u}{\partial x_0^u} p(x_0,t_0|x,t)$$

For drift and diffusion $B_1 = A(x,t)$; $B_2 = D(x,t)$; $B_u = 0$ $u>2$

$$\boxed{-\frac{\partial}{\partial t_0} p(x_0 t_0|x,t) = A(x,t)\frac{\partial}{\partial x_0} p(x_0 t_0|x,t) + \frac{D(x)}{2!}\cdot\frac{\partial^2}{\partial x_0^2} p(x_0 t_0|x,t)}$$

compare to:
forward